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Facility Layout Problems: Solution Approaches and Practical Difficulties

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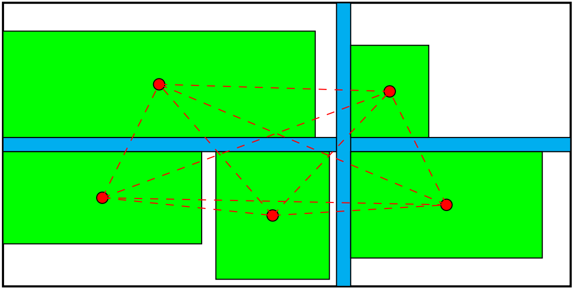
Outline

1. The single row facility layout problem (SRFLP)
2. LP, SDP and Lagrangian relaxation
3. LP-based approaches
4. Semidefinite relaxations
5. An efficient algorithmic approach

Facility Layout Planning

- find an optimal placement of machines inside a factory according to a given objective function



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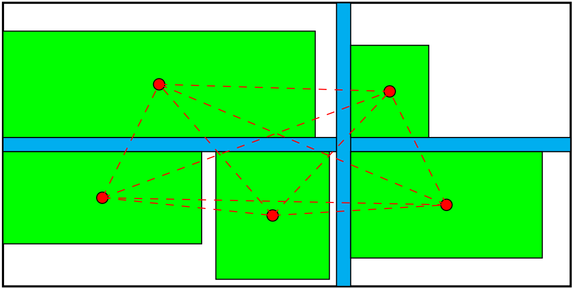

- applications:
  - VLSI circuit design
  - manufacturing systems
  - ...
- very hard problem in general

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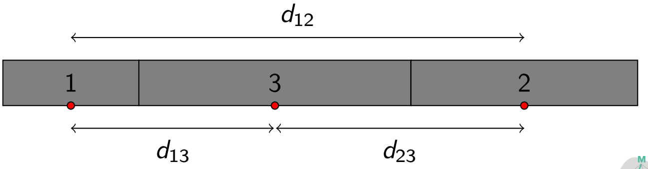
Single Row Facility Layout Problem (SRFLP)

Given:

- n one-dimensional machines $[n] := \{1, \ldots, n\}$
- lengths $\ell_i \geq 0$, $i \in [n]$
- pairwise transport weights $c_{ij} \geq 0$, $i, j \in [n]$, $i < j$

Goal: find a permutation $\pi \in \Pi_n$ of the machines minimizing the total weighted sum of center-to-center distances d_{ij}^{π} between all pairs of machines:

$$\min_{\pi \in \Pi_n} \sum_{\substack{i,j \in [n] \\ i<j}} c_{ij} d_{ij}^{\pi}$$



Literature review

- first considered by Simmons (1969)
- many applications were identified
- many heuristic approaches in recent years
- exact solution methods include: B&B, MILP, DP, ILP, SDP

Best exact solution methods:

- Amaral (2009): Integer Linear Programming ($n \leq 35$)
- Hungerländer & Rendl (2013): semidefinite relaxations ($n \leq 42$)

Related problems:

- equidistant SRFLP is a special case of the QAP
- SRFLP generalizes the (weighted) Linear Arrangement Problem
- other facility layout or ordering problems

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How can we solve a combinatorial optimization problem?

- by enumeration of all possible solutions
- by a suitable combinatorial algorithm
- by computing dual bounds (using mathematical programming)

Solving an instance of the SRFLP to optimality requires two things:

- a feasible solution with some objective value k (upper bound)
- a prove that the optimal value is at least k (lower bound)

$\hookrightarrow$ linear and semidefinite relaxations

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(Mixed-Integer) Linear Programming

Let $c, x \in \mathbb{R}^n$, $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^m$.

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \\ & x_i \in \mathbb{Z}, \quad i \in \mathcal{I} \end{aligned} \tag{MILP}$$

Linear relaxation:

$$\begin{aligned} \min \quad & c^\top x \\ \text{s.t.} \quad & Ax = b \\ & x \geq 0 \end{aligned} \tag{LP}$$

- $\text{opt(LP)} \leq \text{opt(MILP)}$
- (LP) can be solved in polynomial time
- we can also add inequality constraints or free variables

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Semidefinite Programming (SDP)

- $\mathcal{S}_n := \{A \in \mathbb{R}^{n \times n} \colon A = A^\top\}$
- $\langle A, B \rangle := \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$ for any $A, B \in \mathcal{S}_n$

Let $C, A_1, \ldots, A_m \in \mathcal{S}_n$ and $b \in \mathbb{R}^m$. A semidefinite program in standard form can be written as

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \langle A_i, X \rangle = b_i, \quad i = 1, \ldots, m \\ & X \succeq 0. \end{aligned} \tag{SDP}$$

- we also write $\mathcal{A}(X) = b$, where $\mathcal{A} \colon \mathcal{S}_n \to \mathbb{R}^m$ is a linear operator of the form

$$\mathcal{A}(X) = \begin{pmatrix} \langle A_1, X \rangle \\ \vdots \\ \langle A_m, X \rangle \end{pmatrix}$$

- adjoint operator: $\mathcal{A}^\top(y) := \mathcal{A}^*(y) = \sum_{i=1}^m y_i A_i$ for all $y \in \mathbb{R}^m$
- well-posed SDPs can be solved in polynomial time

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Lagrangian relaxation

$$\begin{array}{lll} & \min & \langle C, X \rangle \\ (*) & \text{s.t.} & \mathcal{A}(X) = b \quad (\Leftrightarrow \mathcal{A}(X) - b = 0) \\ & & X \in \mathcal{X} \subseteq \mathcal{S}_n \end{array}$$

- assumption: $(*)$ without $\mathcal{A}(X) = b \in \mathbb{R}^m$ much easier to solve
- primal variable X and dual variable $\mu \in \mathbb{R}^m$
- *Lagrangian*: $\mathcal{L}(X; \mu) := \langle C, X \rangle + \mu^\top (\mathcal{A}(X) - b)$
- *dual function*: $f(\mu) := \inf_{X \in \mathcal{X}} \mathcal{L}(X; \mu)$
- *weak duality*: $f(\mu) \leq \langle C, X \rangle$ for all X feasible in $(*)$ and all $\mu \in \mathbb{R}^m$, since $\mu^\top (\mathcal{A}(X) - b) = 0$ for all X feasible in $(*)$
- dual problem:

$$\begin{array}{ll} \sup & f(\mu) \\ \text{s.t.} & \mu \in \mathbb{R}^m \end{array}$$

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MILP formulation by Love & Wong (1976) I

Intuitive modelling approach with the following variables:

- ordering variables $u_{ij} \in \{0,1\}$, $i,j \in [n]$, $i \neq j$, with the meaning

$$u_{ij} = \begin{cases} 1, & \text{if machine } i \text{ lies to the left of machine } j \\ 0, & \text{otherwise.} \end{cases}$$

- position variables p_i ($\widehat{=}$ abscissa of the centers), $i \in [n]$, with

$$\frac{\ell_i}{2} \leq p_i \leq M - \frac{\ell_i}{2},$$

  where $M := \sum_{i \in [n]} \ell_i$

- distance variables $d_{ij} \geq 0$, $i,j \in [n]$, $i < j$

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MILP formulation by Love & Wong (1976) II

$$
\begin{aligned}
\min \quad & \sum_{\substack{i,j\in[n]\\ i<j}} c_{ij} d_{ij} && \\
\text{s.t.} \quad & u_{ij} + u_{ji} = 1, && i,j \in [n],\ i<j, \\
& d_{ij} \geq p_i - p_j, && i,j \in [n],\ i<j, \\
& d_{ij} \geq p_j - p_i, && i,j \in [n],\ i<j, \\
& p_i + \tfrac{\ell_i + \ell_j}{2} \leq p_j + M(1-u_{ij}), && i,j \in [n],\ i \neq j, \\
& \tfrac{\ell_i}{2} \leq p_i \leq M - \tfrac{\ell_i}{2}, && i \in [n], \\
& d_{ij} \geq 0, && i,j \in [n],\ i<j, \\
& u_{i,j} \in \{0,1\}, && i,j \in [n],\ i \neq j.
\end{aligned}
$$

Very poor linear relaxation: optimal solution is given by

$$
\begin{aligned}
& d_{ij} := 0, \quad i,j \in [n],\ i<j, \\
& p_i := \max\{\ell_j : j \in [n]\}, \quad i \in [n], \\
& u_{ij} := \tfrac{1}{2}, \quad i,j \in [n],\ i \neq j.
\end{aligned}
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\min \quad & \sum_{\substack{i,j\in[n]\\ i<j}} c_{ij} d_{ij} \\
\text{s.t.} \quad & u_{ij} + u_{ji} = 1, && i,j \in [n],\ i<j, \\
& d_{ij} \geq p_i - p_j, && i,j \in [n],\ i<j, \\
& d_{ij} \geq p_j - p_i, && i,j \in [n],\ i<j, \\
& p_i + \tfrac{\ell_i + \ell_j}{2} \leq p_j + M(1 - u_{ij}), && i,j \in [n],\ i \neq j, \\
& \tfrac{\ell_i}{2} \leq p_i \leq M - \tfrac{\ell_i}{2}, && i \in [n], \\
& d_{ij} \geq 0, && i,j \in [n],\ i<j, \\
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Why distance variables should be avoided

- several incremental improvements, e.g., Amaral (2006, 2008)
- significant by Amaral & Letchford (2013): they solved an instance with $n = 30$ in about one day using the lower bounds within a branch-and-bound approach

However:

- still relatively weak lower bounds
- theoretical evidence that the approach is rather limited
- feasible set depends on the concrete instance
- only a ‘local’ modelling, weak coupling

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Betweenness variables (Amaral, 2009)

$b_{ijk} \in \{0,1\}$, $i,j,k \in [n]$, $|\{i,j,k\}| = 3$, $i < k$, with the meaning

$$b_{ijk} = \begin{cases} 1, & j \text{ lies between } i \text{ and } k \\ 0, & \text{otherwise.} \end{cases}$$

Motivation:

$$d_{ij} = \frac{\ell_i + \ell_j}{2} + \sum_{k \in [n] \setminus \{i,j\}} \ell_k b_{ikj}, \qquad i,j \in [n],\ i < j$$

SRFLP formulation:

$$\begin{aligned} \min \quad & \sum_{\substack{i,j \in [n] \\ i<j}} c_{ij} \sum_{k \in [n] \setminus \{i,j\}} \ell_k b_{ikj} + \sum_{\substack{i,j \in [n] \\ i<j}} c_{ij} \frac{\ell_i + \ell_j}{2} \\ \text{s.t.} \quad & \text{the betweenness variables represent a permutation} \end{aligned}$$

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Betweenness model (Amaral, 2009)

$$
\begin{array}{llcll}
\min & \displaystyle\sum_{\substack{i,j\in[n]\\ i<j}} c_{ij} \sum_{k\in[n]\setminus\{i,j\}} \ell_k b_{ikj} + \sum_{\substack{i,j\in[n]\\ i<j}} c_{ij} \frac{\ell_i+\ell_j}{2} & & & \\
\text{s.t.} & b_{ijk} + b_{ikj} + b_{jik} & = & 1, & i,j,k \in [n],\ i<j<k \\
& b_{ihj} + b_{ihk} + b_{jhk} & \leq & 2, & i,j,k,h \in [n],\ |\{i,j,k,h\}| = 4,\ i<j<k \\
& -b_{ihj} + b_{ihk} + b_{jhk} & \geq & 0, & i,j,k,h \in [n],\ |\{i,j,k,h\}| = 4,\ i<j<k \\
& b_{ihj} - b_{ihk} + b_{jhk} & \geq & 0, & i,j,k,h \in [n],\ |\{i,j,k,h\}| = 4,\ i<j<k \\
& b_{ihj} + b_{ihk} - b_{jhk} & \geq & 0, & i,j,k,h \in [n],\ |\{i,j,k,h\}| = 4,\ i<j<k \\
& b_{ijk} \in \{0,1\}, & & & i,j,k \in [n],\ |\{i,j,k\}| = 3,\ i<k
\end{array}
$$

Up to symmetry, there are three cases:

$$
\begin{array}{ccccccc}
\ldots & i & \ldots & j & \ldots & k & \ldots \\
\ldots & i & \ldots & k & \ldots & j & \ldots \\
\ldots & j & \ldots & i & \ldots & k & \ldots
\end{array}
$$

h can only lie between zero **or** two pairs of (i,j), (i,k), (j,k)!

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Properties of the betweenness formulation

Strengths:

- ‘global’ modelling, strong coupling
- linear relaxation often yields the optimal value
- additional inequalities known

Weaknesses:

- simplex method extremely slow
- linear relaxation can already be insufficient for $n = 6$

How can we find even better lower bounds?
$\hookrightarrow$ semidefinite programming (SDP)!

Bivalent quadratic formulation I

Again we use **ordering variables**, but now with values in $\{-1, 1\}$:

$$x_{ij} = \begin{cases} +1, & \text{if } i \text{ lies to the left of } j \\ -1, & \text{otherwise} \end{cases}, \quad i, j \in [n],\ i \neq j.$$

Connection to betweenness variables:

$$b_{ijk} = \frac{1 - y_{ji}y_{jk}}{2},\ j < i < k, \quad b_{ijk} = \frac{1 + y_{ij}y_{jk}}{2},\ j < i < k,$$
$$b_{ijk} = \frac{1 - y_{ij}y_{kj}}{2},\ j < i < k.$$

The following *three-cycle-equations* must be satisfied:

$$x_{ij}x_{jk} - x_{ij}x_{ik} - x_{ik}x_{jk} = -1, \quad i, j, k \in [n],\ i < j < k.$$

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Bivalent quadratic formulation II

$$\min \quad K - \sum_{\substack{i,j\in[n]\\ i<j}} \frac{c_{ij}}{2} \left(\sum_{\substack{k\in[n]\\ k<i}} \ell_k x_{ki} x_{kj} - \sum_{\substack{k\in[n]\\ i<k<j}} \ell_k x_{ik} x_{kj} + \sum_{\substack{k\in[n]\\ k>j}} \ell_k x_{ik} x_{jk} \right)$$

$$\begin{aligned} \text{s.t.} \quad & x_{ij}x_{jk} - x_{ij}x_{ik} - x_{ik}x_{jk} = -1, && i,j,k \in [n],\ i<j<k, \\ & x_{ij} \in \{-1,1\}, && i,j \in [n],\ i<j. \end{aligned}$$

Consider the matrix $X = xx^\top$ with entries $X_{ij,kl} = x_{ij}x_{kl}$. We have:

- $X_{ij,ij} = 1$; we write $\mathrm{diag}(X) = e = (1,\ldots,1)^\top$
- $\mathrm{rk}(X) = 1$
- $X \succeq 0$, since $X = X^\top$ and $v^\top X v = v^\top x x^\top v = (v^\top x)^2 \geq 0$

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Bivalent quadratic formulation II

$$\min \quad K - \sum_{\substack{i,j\in[n]\\ i<j}} \frac{c_{ij}}{2}\left(\sum_{\substack{k\in[n]\\ k<i}} \ell_k x_{ki}x_{kj} - \sum_{\substack{k\in[n]\\ i<k<j}} \ell_k x_{ik}x_{kj} + \sum_{\substack{k\in[n]\\ k>j}} \ell_k x_{ik}x_{jk}\right)$$

$$\begin{aligned} \text{s.t.} \quad & x_{ij}x_{jk} - x_{ij}x_{ik} - x_{ik}x_{jk} = -1, && i,j,k \in [n],\ i<j<k, \\ & x_{ij} \in \{-1,1\}, && i,j \in [n],\ i<j. \end{aligned}$$

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Semidefinite relaxation

Matrix-based formulation:

$$\begin{aligned} \min \quad & \langle C, X \rangle + K \\ \text{s.t.} \quad & X_{ij,jk} - X_{ij,ik} - X_{ik,jk} = -1, \qquad i,j,k \in [n],\ i<j<k, \\ & \operatorname{diag}(X) = e \\ & \operatorname{rk}(X) = 1 \\ & X \succeq 0 \end{aligned}$$

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\end{aligned}
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Strengthened semidefinite relaxation

Anjos et al. (2006), Hungerländer & Rendl (2013) also added the so-called *triangle inequalities*:

$$
\begin{aligned}
\min \quad & \langle C, X\rangle + K \\
\text{s.t.} \quad & X_{ij,jk} - X_{ij,ik} - X_{ik,jk} = -1, \quad i,j,k \in [n],\ i<j<k \\
& \operatorname{diag}(X) = e \\
& X_{ij} + X_{ik} + X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 \\
& X_{ij} - X_{ik} - X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 \\
& -X_{ij} + X_{ik} - X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 \\
& -X_{ij} - X_{ik} + X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 \\
& X \succeq 0
\end{aligned}
\qquad (\mathrm{SDP_{tri}})
$$

Proposition

The semidefinite relaxation ($\mathrm{SDP_{tri}}$) is at least as strong than the linear relaxation of the betweenness model.

- more inequalities: pentagonal inequalities, hexagonal inequalities, heptagonal inequalities, ...

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& \operatorname{diag}(X) = e & \\
& X_{ij} + X_{ik} + X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 & \\
& X_{ij} - X_{ik} - X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 & (\mathrm{SDP_{tri}}) \\
& -X_{ij} + X_{ik} - X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 & \\
& -X_{ij} - X_{ik} + X_{jk} \geq -1, \quad 1 \leq i < j < k \leq n(n-1)/2 & \\
& X \succeq 0 &
\end{array}
$$

Proposition

The semidefinite relaxation (SDP_{tri}) is at least as strong than the linear relaxation of the betweenness model.

- more inequalities: pentagonal inequalities, hexagonal inequalities, heptagonal inequalities, . . .

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How can we solve the semidefinite relaxation?

- we cannot include all inequalities at the same time
- standard interior-point methods require a running time of $\mathcal{O}(n^9)$!

$\hookrightarrow$ we must use a customized, approximative first-order method that is much faster in practice!

We consider the following optimization problem:

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s.t.} \quad & \mathcal{A}(X) \leq a \\ & \mathcal{B}(X) = e \\ & X \succeq 0 \\ & \operatorname{rk}(X) = 1 \Leftrightarrow \|X\|_F^2 = n^2 \end{aligned}$$

Proposition (spherical constraint)

Let $X \in \{Y \in \mathbb{R}^{n \times n} \colon \operatorname{diag}(Y) = e,\ Y \succeq 0\}$. Then we have

$$\|X\|_F \leq n, \qquad \text{and} \qquad \operatorname{rk}(X) = 1 \iff \|X\|_F = n.$$

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Applying Lagrangian relaxation

primal variable: $X \in \mathcal{S}_n$
dual variables: $\lambda \geq 0$, μ, $Z \succeq 0$, α

Lagrangian:

$$\mathcal{L}(X; \lambda, \mu, Z, \alpha) := \langle C, X \rangle + \lambda^\top (\mathcal{A}(X) - a) + \mu^\top (\mathcal{B}(X) - e) + \frac{\alpha}{2} \left(\|X\|^2 - n^2 \right) - \langle Z, X \rangle$$

Dual function:

$$\begin{aligned} f(\lambda, \mu, Z, \alpha) &:= \inf_{X \in \mathcal{S}_n} \mathcal{L}(X; \lambda, \mu, Z, \alpha) \\ &= c(\lambda, \mu, \alpha) + \inf_{X \in \mathcal{S}_n} \left\{ \frac{\alpha}{2} \|X\|^2 + \langle C(\lambda, \mu) - Z, X \rangle \right\}, \end{aligned}$$

where

$$\begin{aligned} c(\lambda, \mu, \alpha) &:= -a^\top \lambda - e^\top \mu - \frac{\alpha}{2} n^2 \\ C(\lambda, \mu) &:= C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu) \end{aligned}$$

Dual problem:

$$\sup_{\lambda \geq 0,\ \mu,\ Z \succeq 0,\ \alpha} f(\lambda, \mu, Z, \alpha)$$

Algorithmic approach I

Theorem (Malick & Roupin (2012))

Given dual variables $\lambda \geq 0$, μ, $Z \succeq 0$ *and* $\alpha > 0$, *the minimum of the Lagrangian* $\mathcal{L}(X; \lambda, \mu, Z, \alpha)$ *is attained at*

$$X = \frac{1}{\alpha}\left(Z - C - \mathcal{A}^\top(\lambda) - \mathcal{B}^\top(\mu)\right)$$

and the dual function can be written

$$f(\lambda, \mu, Z, \alpha) = -a^\top \lambda - e^\top \mu - \frac{\alpha}{2} n^2 - \frac{1}{2\alpha}\left\| C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu) - Z \right\|^2.$$

Theorem (Malick & Roupin (2012))

Given dual variables (λ, μ, α) *with* $\alpha > 0$, *the dual function can be maximized over* Z; *the resulting simplified dual function is*

$$\begin{aligned} f(\lambda, \mu, \alpha) &:= \max_{Z \succeq 0} f(\lambda, \mu, Z, \alpha) \\ &= -a^\top \lambda - e^\top \mu - \frac{\alpha}{2} n^2 - \frac{1}{2\alpha}\left\| \left[C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu) \right]_- \right\|^2, \end{aligned}$$

where $[\,\cdot\,]_-$ *denotes the projection onto the cone of negative semidefinite matrices.*

Algorithmic approach II

- $f(\lambda, \mu, \alpha)$ is differentiable at any (λ, μ, α) with $\alpha > 0$ and the partial derivatives are

$$\partial_\lambda f(\lambda, \mu, \alpha) = -\frac{1}{\alpha}\mathcal{A}\left(\left[C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu)\right]_-\right) - a$$
$$\partial_\mu f(\lambda, \mu, \alpha) = -\frac{1}{\alpha}\mathcal{B}\left(\left[C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu)\right]_-\right) - e$$

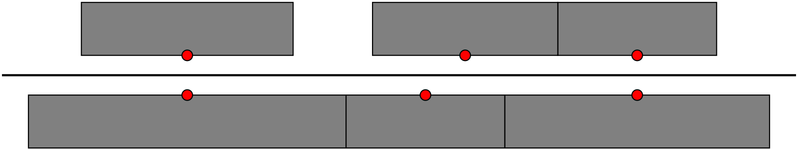
Algorithmic idea:

- fix $\overline{\alpha} > 0$ and optimize $f(\lambda, \mu, \overline{\alpha})$ over $\lambda \geq 0$ and μ
- in the SRFLP setting: by taking $\alpha > 0$ small enough, we can get arbitrarily close to the bound of the semidefinite relaxation

Results:

- outperforms all approaches in the literature (faster, stronger bounds)
- by using additional pentagonal, hexagonal and heptagonal inequalities, SRFLP instances with up to $n = 81$ could be solved for the first time

The Double Row Facility Layout Problem (DRFLP)



- solution is no permutation
- gaps are possible
- distances may be zero
- can distance variables be avoided?
- is there any good semidefinite relaxation?
- if yes, how can it be solved?

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