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A Semidefinite Approach for the Single Row Facility Layout Problem

FWF Der Wissenschaftsfonds.

UNIVERSITÄT KLAGENFURT

Modeling - Analysis - Optimization DOC.FUNDS DOCTORAL SCHOOL

Outline

- Master's thesis: "Solution Approaches for the Single Row Facility Layout Problem based on Semidefinite Programming"
- TU Dortmund
- Supervisor: Prof. Dr. Anja Fischer

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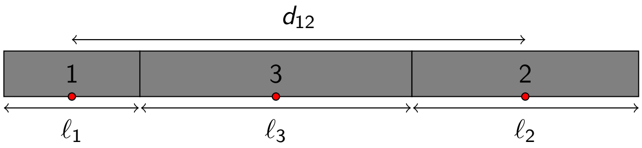

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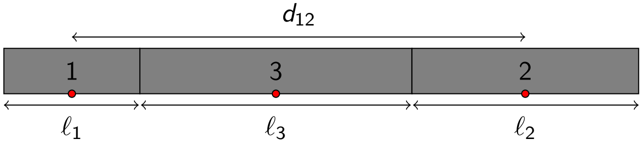

- strongly $\mathcal{NP}$-hard
- many applications (e.g., in manufacturing systems)

Betweenness Approach (Amaral, 2009)

Betweenness variables:

$$b_{ikj} = \begin{cases} 1, & \text{if } k \text{ lies between } i \text{ and } j \\ 0, & \text{otherwise} \end{cases}, \quad k \neq i < j \neq k$$

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- simplex method impractical

Semidefinite Formulation

$$\textbf{Ordering variables: } x_{ij} = \begin{cases} +1, & \text{if } i \text{ left of } j \\ -1, & \text{otherwise} \end{cases}, \quad i,j \in [n],\ i < j$$

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Up to a constant, (SRFLP) is equivalent to (see Anjos et al., 2005)

$$\min \quad \sum_{\substack{i,j\in[n] \\ i<j}} \frac{c_{ij}}{2} \left(-\sum_{\substack{k\in[n] \\ k<i}} \ell_k x_{ki} x_{kj} - \sum_{\substack{k\in[n] \\ i<k<j}} \ell_k x_{ik} x_{kj} + \sum_{\substack{k\in[n] \\ k>j}} \ell_k x_{ik} x_{jk} \right)$$

$$\begin{aligned} \text{s.t.} \quad & x_{ij}x_{jk} - x_{ij}x_{ik} - x_{ik}x_{jk} = -1, && i,j,k \in [n],\ i<j<k, \quad (*) \\ & x_{ij} \in \{-1, 1\}, && i,j \in [n],\ i<j. \end{aligned}$$

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Semidefinite lifting up to the symmetric matrix space:

$$\min \left\{ \langle C, X \rangle : X \text{ satisfies } (*),\ \operatorname{diag}(X) = e,\ X \succeq 0,\ \operatorname{rank}(X) = 1 \right\},$$

where $X = xx^\top$ with entries $X_{ij,kl} = x_{ij}x_{kl}$.

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- $\mathcal{O}(n^6)$ triangle inequalities can be added as cutting planes:

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X_{i,j} + X_{i,k} + X_{j,k} &\geq -1, & 1 \leq i < j < k \leq \tbinom{n}{2} \\
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- interior-point methods (IPMs) require $\mathcal{O}(n^9)$ time to solve (SDP_0)
- Hungerländer & Rendl (2012, 2013):
  - additional 'matrix cuts'
  - partial Lagrangian approach
  - IPMs + bundle method $\hookrightarrow$ nonsmooth optimization
  - instances with $n \leq 42$ solved

Strengthened Relaxation

- $\mathcal{O}(n^{10})$ pentagonal inequalities $\sum_{1 \leq i < j \leq 5} \delta_i \delta_j X_{p_i, p_j} \geq -2, \quad \delta_k \in \{\pm 1\}$

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$$(i, r), (j, r), (k, r), (l, r), (m, r)$$

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Proposition

The semidefinite relaxation ($SDP_{\mathcal{P}^*}$) is at least as strong as the linear relaxation of the betweenness approach.

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- heuristic separation for general pentagonal, hexagonal, and heptagonal inequalities

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$$\begin{aligned} \sup \quad & \left\{-e^\top\lambda - e^\top\mu - \frac{\alpha}{2}\binom{n}{2}^2 - \frac{1}{2\alpha}\left\|\left[C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu)\right]_-\right\|_F^2\right\} \\ \text{s.t.} \quad & \lambda \ge 0,\ \mu \text{ free} \end{aligned} \qquad (\text{DSDP}_\alpha)$$

- $\mathcal{A}^\top(\cdot),\ \mathcal{B}^\top(\cdot)$: adjoint operators
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- $\|\cdot\|_F$: Frobenius norm
- (DSDP_α) convex optimization problem with bound constraints
- objective function differentiable $\hookrightarrow$ L-BFGS-B method
- usual SDP bound can be approximated with arbitrary precision

Algorithmic Approach II

Cutting plane approach:

- L-BFGS-B method can be warm-started
- $X_{\text{approx}} = -\frac{1}{\alpha}\left[C + \mathcal{A}^\top(\lambda) + \mathcal{B}^\top(\mu)\right]_-$

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Implementation:

- C implementation
- BiqCrunch as template

Results and Future Work

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Future work:

- branch-and-bound approach

References

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