September 4, 2024

A low-rank high-precision solver for semidefinite programming

Joint work with Daniel Brosch and Angelika Wiegele

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UNIVERSITÄT KLAGENFURT

Modeling - Analysis - Optimization DOC.FUNDS DOCTORAL SCHOOL

Semidefinite programming

SDP in standard form

$$\begin{aligned} \min \quad & \langle C, X \rangle \\ \text{s. t.} \quad & \mathcal{A}(X) = b \\ & X \in \mathcal{S}_n^+ \end{aligned} \tag{SDP}$$

- $C, A_1, \ldots, A_m \in \mathcal{S}_n,\ b \in \mathbb{R}^m$, linear operator $\mathcal{A}\colon \mathcal{S}_n \to \mathbb{R}^m$

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- adjoint operator $\mathcal{A}^\top\colon \mathbb{R}^m \to \mathcal{S}_n$ with $\mathcal{A}^\top(y) = \sum_{i=1}^m y_i A_i$

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- adjoint operator $\mathcal{A}^\top \colon \mathbb{R}^m \to \mathcal{S}_n$ with $\mathcal{A}^\top(y) = \sum_{i=1}^m y_i A_i$
- **assumption:** strong duality holds

Motivation

Fixed-precision solvers:

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- `SDPA-GMP`
- `Clarabel`
- `COSMO`
- `ClusteredLowRankSolver`
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- project name: `The Augmented Mixing Method`

Main idea

Burer & Monteiro (2003): `SDPLR` (SDPs in standard form)

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Idea: combine both approaches (Augmented Mixing Method)

- tackle general SDPs
- solve small subproblems with high precision
- ADMM-style method

Burer-Monteiro factorization (Burer & Monteiro, 2003)

Theorem (Barvinok, 1995; Pataki, 1998)

If $\bar{X}$ is an extreme point of (SDP), then $\operatorname{rank}(\bar{X}) \leq k_m$, *where* $k_m := \max\{k \in \mathbb{N} \colon k(k+1)/2 \leq m\}$.

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Change of variables: $X = V^\top V$

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- Barvinok-Pataki bound: $k \geq \sqrt{2m}$

Non-convex reformulation

Reformulation

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Proposition (Burer & Monteiro, 2005)

Suppose $X = V^\top V$ is feasible for (LR-SDP). Then X is a local minimum of (LR-SDP) if and only if V is a local minimum of $(*)$.

Augmented Lagrangian approach

Augmented Lagrangian

$$\mathcal{L}(V, y; \mu) := \langle C, V^\top V\rangle + \frac{\mu}{2}\|b - \mathcal{A}(V^\top V)\|^2 + \langle y, b - \mathcal{A}(V^\top V)\rangle$$

- penalty parameter $\mu > 0$
- vector of Lagrange multipliers $y \in \mathbb{R}^m$

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Derivative w.r.t V

$$\nabla_V \mathcal{L}(V, y; \mu) = 2V\tilde{S}$$

where

$$\tilde{S} = C - \sum_{i=1}^{m} \tilde{y}_i A_i$$

$$\tilde{y} = y - \mu(\mathcal{A}(V^\top V) - b)$$

Implementation of augmented Lagrangian approach

Algorithm 1 SDPLR (Burer & Monteiro, 2003)

1. Choose starting values V^0, y^0, μ^0, and set $p := 0$.
2. While $\|b - \mathcal{A}(V^{p\top} V^p)\|$ too large:
   - Compute $V^{p+1} := \arg\min_{V \in \mathbb{R}^{k \times n}} \mathcal{L}(V^p, y^p; \mu^p)$.
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     Otherwise:
     - Set $y^{p+1} := y^p$.
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- primal method in kn variables
- quasi-Newton method used
- no eigenvalue computations, exploits sparsity

Theoretical results

Max-Cut relaxation

$$\begin{aligned} \max \quad & \langle C, X \rangle \\ \text{s. t.} \quad & X_{ii} = 1, \; i = 1, \ldots, n \\ & X \in \mathcal{S}_n^+ \end{aligned} \tag{MC-SDP}$$

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However: strong practical performance with smaller values of k
Issue: cannot achieve very high accuracy in many cases

Burer-Monteiro factorization for Max-Cut

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Column-wise storage

$X = V^\top V \succeq 0, \; V = (v_1 | \dots | v_n) \in \mathbb{R}^{k \times n}$:

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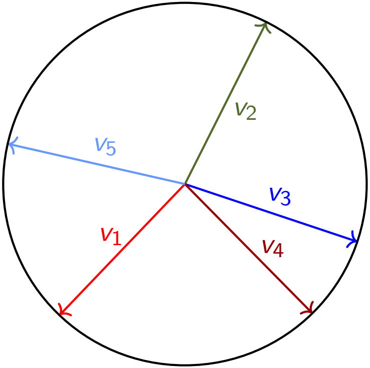
- Barvinok-Pataki bound: (MC-SDP) $\Leftrightarrow$ (MC-vec) for $k \geq \sqrt{2n}$

Geometric interpretation

Optimization problem (MC-vec)

$$\max \quad \langle C, V^\top V \rangle = \sum_{i,j=1}^{n} C_{ij} v_i^\top v_j \qquad \text{(MC-vec)}$$
$$\text{s. t.} \quad \|v_i\| = 1, \; i = 1, \ldots, n$$



$$v_i^\top v_j = \|v_i\| \cdot \|v_j\| \cdot \cos \measuredangle(v_i, v_j)$$
$$= \cos \measuredangle(v_i, v_j)$$

Coordinate ascent approach

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We fix all columns except v_i. (MC-vec) reduces to

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where $g = \sum_{j=1, j \neq i}^{n} C_{ij} v_j = V \cdot C_{(i)} - C_{ii} v_i$.

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- closed-form solution: $v_i = \frac{g}{\|g\|}$ if $g \neq 0$

Mixing Method (Wang et al., 2018)

Algorithm 2 Mixing Method (Wang et al., 2018)

Input: $C \in \mathbb{R}^{n \times n}$ with $\operatorname{diag}(C) = 0,\ k \in \mathbb{N}_{\geq 1}$
Output: approximate solution $V = (v_1 | \dots | v_n) \in \mathbb{R}^{k \times n}$ of (SDP-vec)
for $i \leftarrow 1$ **to** n **do**
    $v_i \leftarrow$ random vector on the unit sphere $\mathcal{S}^{k-1}$
end
while *not yet converged* **do**
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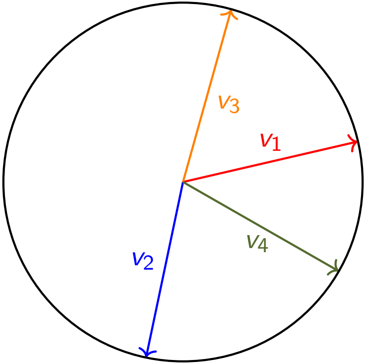
Theorem: local linear convergence (Wang et al., 2018)

Let $k > \sqrt{2n}$. If the iterates do not degenerate, then the Mixing Method converges locally to the global optimum of (SDP-vec) at a linear rate.

Example with $n = 4$ and $k = 2$

$$C = \frac{1}{4} \begin{pmatrix} 1 & 1 & 1 & -3 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -4 & 2 \\ -3 & -1 & 2 & 2 \end{pmatrix}$$



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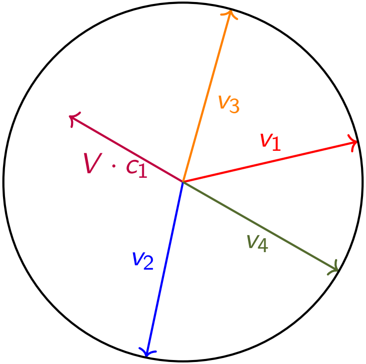

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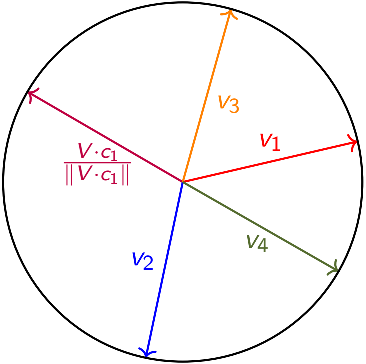

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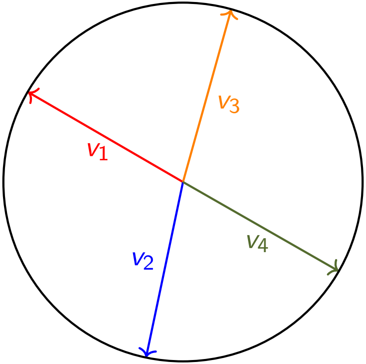

$$\langle C, V^\top V\rangle = 0.0701836938398076$$

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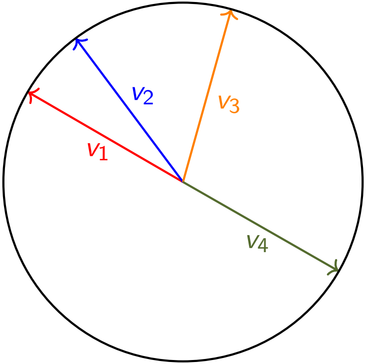

$$\langle C, V^\top V \rangle = 2.1042821481042009$$

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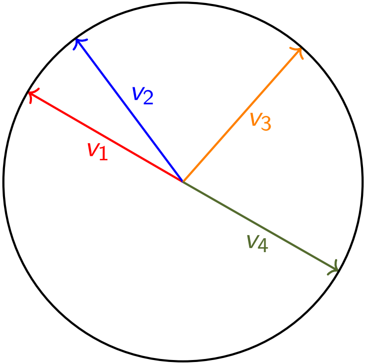

$$\langle C, V^\top V\rangle = 2.1248497956082537$$

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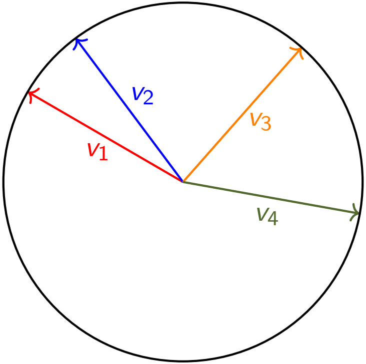

$$\langle C, V^\top V \rangle = 2.2584781813631301$$

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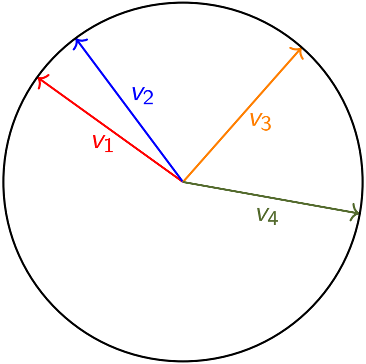

$$\langle C, V^\top V \rangle = 2.2669613535505473$$

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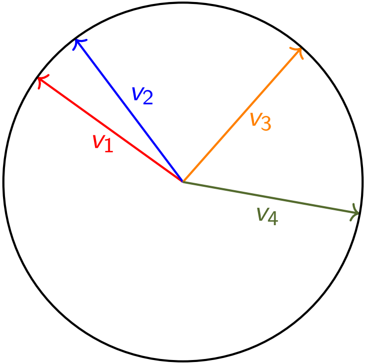

$$\langle C, V^\top V \rangle = 2.2669669930002718$$

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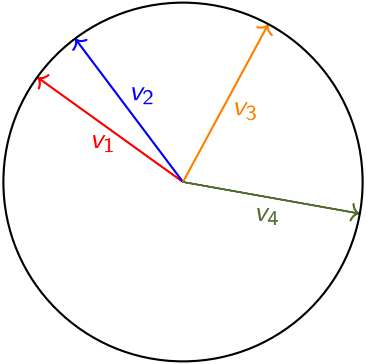

$$\langle C, V^\top V \rangle = 2.2820426702215686$$

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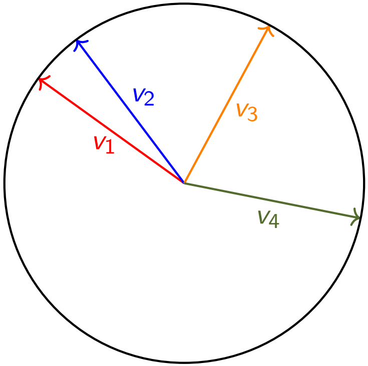

$$\langle C, V^\top V \rangle = 2.2824146853764495$$

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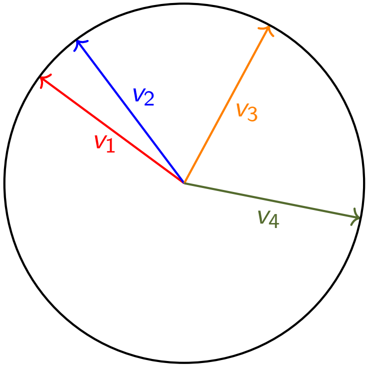

$$\langle C, V^\top V \rangle = 2.2825485984904232$$

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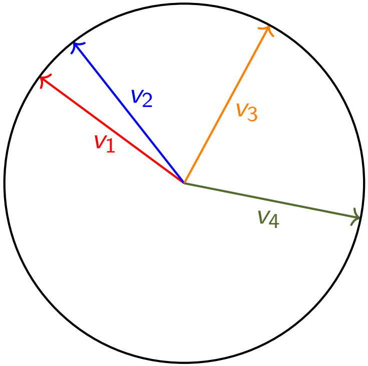

$$\langle C, V^\top V \rangle = 2.2827921992397187$$

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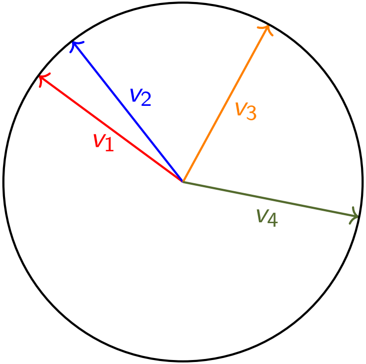

$$\langle C, V^\top V \rangle = 2.2827965824488148$$

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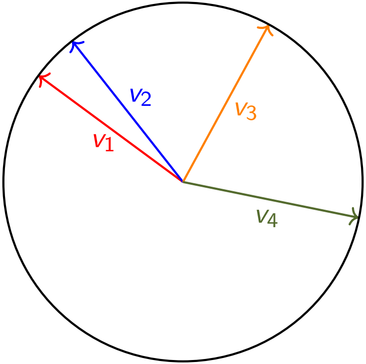

$$\langle C, V^\top V \rangle = 2.2828175664597827$$

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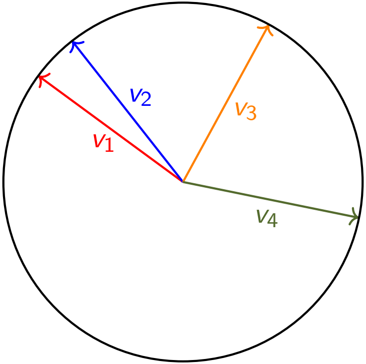

$$\langle C, V^\top V \rangle = 2.2828214514872149$$

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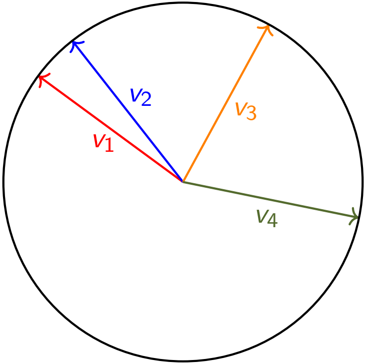

$$\langle C, V^\top V \rangle = 2.2828225671023645$$

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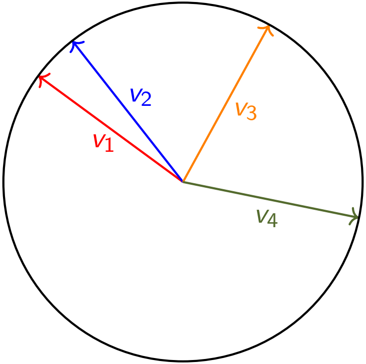

$$\langle C, V^\top V\rangle = 2.2828245614424776$$

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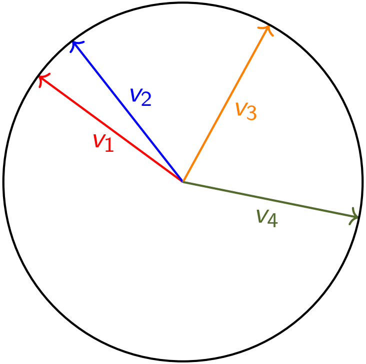

$$\langle C, V^\top V \rangle = 2.2828250454404815$$

The Augmented Mixing Method

Algorithm 3 `SDPLR`

---

1. Choose starting values V^0, y^0, μ^0, and set $p := 0$.
2. While $\|b - \mathcal{A}({V^p}^\top V^p)\|$ too large:
   - Compute $V^{p+1} := \arg\min_{V \in \mathbb{R}^{k \times n}} \mathcal{L}(V^p, y^p; \mu^p)$.
   - If $\|b - \mathcal{A}({V^{p+1}}^\top V^{p+1})\|$ has sufficiently decreased:
     - Update $y^{p+1} := y^p - \mu^p(\mathcal{A}({V^{p+1}}^\top V^{p+1}) - b)$.
     - Set $\mu^{p+1} := \mu^p$.

     Otherwise:
     - Set $y^{p+1} := y^p$.
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   - Set $p := p + 1$.

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- subproblems only have k variables
- can easily be solved with arbitrary precision

The Augmented Mixing Method

Algorithm 3 `Augmented Mixing Method`

1. Get V^0, y^0, μ^0 by warm-starting from `SDPLR`, and set $p := 0$.
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   - **For** $i = 1, \dots, n$ **do**

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Small subproblems

Full augmented Lagrangian

$$\mathcal{L}(V, y; \mu) := \langle C, V^\top V \rangle + \frac{\mu}{2} \| b - \mathcal{A}(V^\top V) \|^2 + \langle y, b - \mathcal{A}(V^\top V) \rangle$$

Small subproblems

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Subproblem

$$\min_{v_i \in \mathbb{R}^k} \mathcal{L}(V^p, y^p; \mu^p)$$

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Subproblem

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Derivative w.r.t v_i

$$\frac{\partial}{\partial v_i}\mathcal{L}(V, y; \mu) = 2V\tilde{S}_{(i)}$$

where

$$\tilde{S} = C - \sum_{i=1}^{m} \tilde{y}_i A_i$$

$$\tilde{y} = y - \mu(\mathcal{A}(V^\top V) - b)$$

Implementation

Standard stopping criteria for SDPs

For $X, Z \in \mathcal{S}_n^+$ and $y \in \mathbb{R}^m$:

$$\texttt{pinf} := \frac{\|\mathcal{A}(X) - b\|_2}{1 + \|b\|_2} < \texttt{tol}$$

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- `SDPA-GMP` needs roughly one minute for $(n, m) = (20, 40)$

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